

Theory and Applications of Fourier, Laplace, and Z Transformations

Carlo Ciulla

Epoka University, Faculty of Architecture and Engineering, Department of Computer Engineering,
Rr. Tiranë-Rinas, Km. 12 - 1032 Vorë, Tirana, Albania

E-mail: cciulla@epoka.edu.al, cxc2728@njit.edu

Abstract: This study presents the mathematics for the implementation of direct and inverse Fourier, Laplace, and Z transformations. This research is at the intersection between signal processing, applied mathematics, and software engineering, and it provides a study guide to implementers. Mathematical concepts and details necessary to transform the math into code are provided as theoretical background. Validation is conducted for the cases when the transforms do intersect, when the transforms do not intersect, and when, in Fourier and Z-transformations, the frequency domain encodes a phase shift which is reconstructed as an image space shift. Coherence between the software implementation of the three transformations is confirmed when: 1. The real component σ of the complex variable $s = \sigma + i\omega$ is equal to zero, which is the case when Fourier and Laplace transforms are the same. 2. When the magnitude r of the complex variable $z = r e^{j\omega}$ is equal to one, which is the case when Fourier and Z transforms are the same. Congruency between software implementation of transformations is confirmed comparing departing image and inverse reconstructed image. The novelty of this research is the presentation style of the theory of direct and inverse Fourier, Laplace, and Z transforms. Details provided in this research make this paper a study guide that is not found elsewhere.

Keywords: k-space, L-space, z-space, phase, shift.

Received 13 June 2022

Accepted 20 July 2022

Published 26 July 2022

1. Introduction

1.1. Historical perspective & research objective

The effective and time-saving use of the Fourier transformation had triggered in the past the development of fast Fourier transform, which in its pioneer works benefit of Cooley and Tukey's breakthrough [1, 2, 3], which also facilitated the calculation of Laplace and Z transforms [4]. Numerical Laplace inversion algorithms were reported with a focus on automatic digital computation [5] and algorithmic notation [6]. The relationship between Magnetic Resonance Imaging (MRI) and basic Fourier principles were revealed by Peter Mansfield [7] & Paul Lauterbur [8]. The state of the art in the literature provides well-known and well-established theoretical concepts in relation to direct and inverse Fourier, Laplace, and Z transformations. Nevertheless, the challenge to implement into functional code the transformations, paves the path to a learning experience that starts from the theoretical background and through empirics verifies the correctness of software implementation. Indeed, pioneer work on the numerical inversions of Laplace transforms was carried out long ago by Weeks [9] and Talbot [10]. More recent work provides a description of software implementations of numerical

inversion of a Laplace transform [11-13] and reviews the literature on inverse Laplace transform algorithms [14]. The Laplace transform can be used to solve nonlinear ordinary partial differential equations [15] and fractional differential equations [16]. One aspect of this research is indeed the validation of software implementation of inverse Laplace transform [11, 12, 13]. To this purpose, this paper addresses in the result section a validation paradigm that clarifies, for the dataset of two-dimensional images, when the implementation of the inverse Laplace transform is successful. After Laplace, the basic knowledge of the Z-transform was reintroduced by the Polish mathematician Witold Hurewicz in 1947. The specificity of the Z-transform to model sample data was studied by Ragazzini and Zadeh [17] and later characterized by Jury [18]. Both in nature and in engineering, phenomena can be modelled through differential equations which can determine closed-form solutions [19]. To the extent of the generation of closed-form solutions, Laplace and Z transforms can be successful [14, 19]. More specifically, under a theoretical perspective, the continuity required by Fourier and Laplace is released by the Z-transform allowing the latter to be used to study phenomena regulated by a set of discrete samples while relinquishing the approximation that would be required if Fourier and Laplace transforms would be used for modelling [19, 20]. As far as regards the three transforms treated in this paper, computational evolutions reported in the literature include:

- Fractional Fourier transform [21];



© The Author(s). Published by CIEES Association Press. This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivs 4.0 Generic License.

- Numerical methods for the inversion into the real domain of solutions of differential equations defined in the Laplace domain [22];
- The chirp Z-transform [23].

However, the task of implementing the theory of Fourier, Laplace, and Z transforms is not trivial neither is the validation of the implementation. The motivation of this work is to generate mathematical theory that facilitate code implementation of the three direct and inverse transformations. The generation of code necessitates the translation of the concepts into practical formulations which simplify the implementation. The contribution of this research is hence in the details of theory which are often omitted in the literature to support the introduction of more advanced concepts.

1.2. Validation of the theory

There are two approaches undertaken by this research to validate the theory.

The first approach works through intersections. The Laplace transform uses the complex variable $s = \sigma + i \omega$. When $s = i \omega$, which means when $\sigma = 0$, the Laplace transform is the Fourier transform. Similarly, the intersection between Z and Fourier transformations ensues when $r = 1$ for the complex variable $z = r e^{j\omega}$. These aspects are thought of as validation evidence. Validation of the intersection between direct and inverse Fourier and Laplace transformations, and validation of the intersection between direct and inverse Fourier and Z transformations; are evidence of correct implementation for these specific cases.

The second approach is the implementation of the inverse transformations so to pursue validation of direct transformations. This validation approach is used to collect evidence of correct implementation when the intersections are not in place. Which means when $\sigma \neq 0$ (Laplace transforms) and when $r \neq 1$ (Z transforms). In this paper the frequency domain calculated with Fourier, Laplace, and Z transforms is termed as k-space, L-space, and z-space respectively.

2. Theory

2.1. Direct Fourier transformation

The direct Fourier transformation [24] is

$$f(\varphi) = \int_{-\infty}^{+\infty} f(z) e^{-2\pi i \varphi z} dz \quad (1)$$

The phase is $\varphi = 2.0 \pi x y / (N_x \cdot N_y)$, z is the complex variable, $f(z)$ is the signal represented as a complex number, $f(\varphi)$ is the frequency or so-called k-space. The phase is the unique identifier of the signal component. In Fourier processing and more specifically in Fourier direct and inverse transformations, the phase is like a label to use to map the signal. The digital signal can be viewed as a sequence of samples and each sample can be identified through the phase. Each signal component (sample) is assigned to a different value of phase φ and so it can be identified. The coordinate (x, y) localizes the pixel of the image to be processed (labelled).

Given the integration between $-\infty$ and $+\infty$, In addition to what reported in the appendix, Eq. (1) requires the following elaboration. We make the approximation that

the signal is discrete and finite. Discrete because represented by a set of samples which we call pixel intensity values (or brightness values). Finite because the signal exists between 1 and $(N_x \cdot N_y)$.

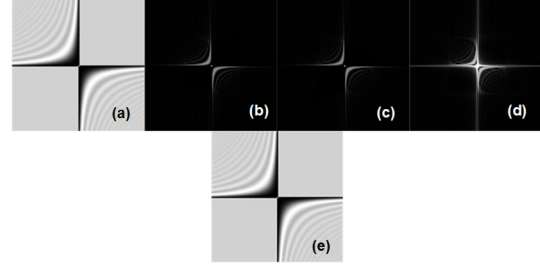


Fig. 1. Sinc signal image (a), the real component of the k-space (b), the imaginary component of the k-space (c), the k-space magnitude (d). Reconstructed image by inverse Fourier Transformation (e).

Eq. (1) asserts that the signal $f(z)$ can be decomposed into a convolution of signal intensity values and cosine and sine waves. More specifically: when $f(z)$ is a real number (pixel brightness), that is $f_k(z)$ (with $k = 1, 2 \dots (N_x \cdot N_y)$); and when $2 \pi \varphi z = \text{phase} = 2.0 \pi x y / (N_x \cdot N_y)$; from Eq. (1) we can write that the s -th real component of the k-space is (F_s, Re) , and the s -th imaginary component of the k-space is (F_s, Im) .

$$(F_s, \text{Re}) + i (F_s, \text{Im}) \approx$$

$$\sum_{k=1}^{(N_x \cdot N_y)} f_k(z) \cdot [\cos(\text{phase}_{ks}) + \sin(\text{phase}_{ks})] + \sum_{k=1}^{(N_x \cdot N_y)} i f_k(z) \cdot [-\sin(\text{phase}_{ks}) + \cos(\text{phase}_{ks})] \quad (2)$$

with $s = 1, 2 \dots (N_x \cdot N_y)$

The value of the phase changes for each (k, s) couple determined by $k = 1, 2 \dots (N_x \cdot N_y)$, and $s = 1, 2 \dots (N_x \cdot N_y)$. Practically, the value of the phase is implemented as $2.0 \pi k_2 k_3 / (N_x \cdot N_y)$, and it changes at each iteration of four nested for loops (two outer loops update the value of k_2 , and two innermost loops update the value of k_3). The value of the phase is thus always different at each iteration. Thus, k_2 is the same when all of the values of k_3 span in $[1, 2 \dots (N_x \cdot N_y)]$ (k_3 changes within two for loops selecting each image pixel intensity value). After that, the value of k_2 changes (to target another k-space pixel) and k_3 spans again in $[1, 2 \dots (N_x \cdot N_y)]$. The same concept of the phase change applies to inverse Fourier transform and direct and inverse Laplace, and Z transforms. Now let us implement this formula to calculate the direct Fourier transformation. The equation of the s -th pixel of the real component of the k-space is

$$(F_s, \text{Re}) =$$

$$\sum_{k=1}^{(N_x \cdot N_y)} f_k(z) \cdot [\cos(\text{phase}_{ks}) + \sin(\text{phase}_{ks})] \quad (3)$$

with $s = 1, 2 \dots (N_x \cdot N_y)$

The equation of the s -th pixel of the imaginary component of the k-space is

$$(F_s, \text{Im}) =$$

$$\sum_{k=1}^{(N_x \cdot N_y)} f_k(z) \cdot [-\sin(\text{phase}_{ks}) + \cos(\text{phase}_{ks})] \quad (4)$$

with $s = 1, 2 \dots (N_x \cdot N_y)$

Given the departing image in Figure 1a, (F_s, R_e) is the s -th pixel of the real component image of the k-space (see Figure 1b), (F_s, I_m) is s -th pixel of the imaginary component image of the k-space (see Figure 1c). This means that each pixel (F_s, R_e) of the real component image of the k-space is calculated with Eq. (3) for $s = 1, 2 \dots (N_x \cdot N_y)$. Also, this means that each pixel (F_s, I_m) of the imaginary component image of the k-space is calculated with Eq. (4) for $s = 1, 2 \dots (N_x \cdot N_y)$. Recall that k-space means frequency domain. The frequency domain is also characterized by the k-space magnitude $M_s = \sqrt{(F_s, \text{Re})^2 + (F_s, \text{Im})^2}$ with $s = 1, 2 \dots (N_x \cdot N_y)$ (see Figure 1d).

2.2 Inverse Fourier transformation

The inverse Fourier transformation [24] is

$$f(z) = \int_{-\infty}^{+\infty} f(\varphi) e^{2\pi i \varphi z} d\varphi \quad (5)$$

Now, in addition to what reported in the appendix, let us consider that $[(F_s, R_e) + i (F_s, I_m)]$ is the notation of the s -th term of the k-space (made of real and imaginary component), with $s = 1, 2 \dots (N_x \cdot N_y)$. As we are in the discrete domain, we are processing a finite signal. Therefore, the integration between $-\infty$ and $+\infty$ can be replaced by the finite sum. Eq. (5) yields

$$\sum_{s=1}^{(N_x \cdot N_y)} [(F_s, \text{Re}) \cdot \cos(\text{phase}_{sj}) - (F_s, \text{Im}) \cdot \sin(\text{phase}_{sj})] + \sum_{s=1}^{(N_x \cdot N_y)} i [(F_s, \text{Re}) \cdot \sin(\text{phase}_{sj}) + (F_s, \text{Im}) \cdot \cos(\text{phase}_{sj})] \quad (6)$$

with $j = 1, 2 \dots (N_x \cdot N_y)$

The value of the phase changes for each (s, j) couple determined by $s = 1, 2 \dots (N_x \cdot N_y)$, and $j = 1, 2 \dots (N_x \cdot N_y)$. The inverse Fourier transformation uses the k-space to transform back into the departing image (see Figure 1e). Eq. (6) indicates that the j -th pixel of the real component of the reconstructed signal is

$$\sum_{s=1}^{(N_x \cdot N_y)} [(F_s, \text{Re}) \cdot \cos(\text{phase}_{sj}) - (F_s, \text{Im}) \cdot \sin(\text{phase}_{sj})] \quad (7)$$

with $j = 1, 2 \dots (N_x \cdot N_y)$

And the j -th pixel of the imaginary component of the reconstructed signal is

$$\sum_{s=1}^{(N_x \cdot N_y)} [(F_s, \text{Re}) \cdot \sin(\text{phase}_{sj}) + (F_s, \text{Im}) \cdot \cos(\text{phase}_{sj})] \quad (8)$$

with $j = 1, 2 \dots (N_x \cdot N_y)$

And the j -th pixel of the magnitude of the reconstructed signal is

$$f_j(z) = \sqrt{(f_j(z), \text{Re})^2 + (f_j(z), \text{Im})^2} \quad (9)$$

with $j = 1, 2 \dots (N_x \cdot N_y)$

2.3 Direct Laplace transformation

The direct Laplace transformation [24] is

$$f(s) = \int_{-\infty}^{+\infty} f(z) e^{-sz} dz \quad (10)$$

The complex variable is $s = \sigma + i \omega$. When $s = i \omega$, which means when $\sigma = 0$

$$f(s) = \int_{-\infty}^{+\infty} f(z) e^{-sz} dz = \int_{-\infty}^{+\infty} f(z) e^{-i\omega z} dz \quad (11)$$

Eq. (11) is the Fourier transform of $f(z)$. The Laplace transform also bears a straightforward relationship to the Fourier transform when the complex variable is not purely imaginary. To see this relationship, consider $f(s)$ with s expressed as $s = \sigma + i \omega$, so that

$$f(s) = \int_{-\infty}^{+\infty} f(z) e^{-(\sigma + i \omega)z} dz = \int_{-\infty}^{+\infty} [f(z) e^{-\sigma z}] e^{-i\omega z} dz \quad (12)$$

We recognize the right-hand side of Eq. (12) as the Fourier transform of $[f(z) e^{-\sigma z}]$, which means that the Laplace transform of $f(z)$ can be interpreted as the Fourier transform of $f(z)$ after multiplication by a real exponential signal $e^{-\sigma z}$ [24]. Note that $\sigma \cdot z$ can be called ρ , so that we use the position: $e^{-\rho}$. Moreover, when $2\pi \varphi z = \text{phase} = 2.0 \pi x y / (N_x \cdot N_y)$, from Eq. (12) it follows that

$$\sum_{k=1}^{(N_x \cdot N_y)} f_k(z) \cdot [\cos(\text{phase}_{kj}) + \sin(\text{phase}_{kj})] \cdot e^{-\rho} + \sum_{k=1}^{(N_x \cdot N_y)} i f_k(z) \cdot [-\sin(\text{phase}_{kj}) + \cos(\text{phase}_{kj})] \cdot e^{-\rho} \quad (13)$$

with $j = 1, 2 \dots (N_x \cdot N_y)$

The value of $f_k(z)$ is the k -th image pixel brightness. The value of the phase changes for each (k, j) couple determined by $k = 1, 2 \dots (N_x \cdot N_y)$, and $j = 1, 2 \dots (N_x \cdot N_y)$. The j -th pixel of the real component of the L-space (one pixel of the image in Figure 2b) is

$$(f_j(s), \text{Re}) = \sum_{k=1}^{(N_x \cdot N_y)} f_k(z) \cdot [\cos(\text{phase}_{kj}) + \sin(\text{phase}_{kj})] \cdot e^{-\rho} \quad (14)$$

with $j = 1, 2 \dots (N_x \cdot N_y)$

And the j -th pixel of imaginary component of the L-space (one pixel of the image in Figure 2c) is

$$(f_j(s), \text{Im}) = \sum_{k=1}^{(N_x \cdot N_y)} f_k(z) \cdot [-\sin(\text{phase}_{kj}) + \cos(\text{phase}_{kj})] \cdot e^{-\rho} \quad (15)$$

with $j = 1, 2 \dots (N_x \cdot N_y)$

Recall that L-space (or s -space as many textbooks report) means frequency domain. The L-space magnitude is $M_j = \sqrt{(f_j(s), \text{Re})^2 + (f_j(s), \text{Im})^2}$ with $j = 1, 2 \dots (N_x \cdot N_y)$ (see Figure 2d).

The Laplace transform (see Eq. (15)) is defined for $\sigma > \alpha$ (real number) [24]. The evidence of the assertion is presented in Figure 3, where it was determined that $\alpha > -$

5.0. Figure 3 shows the case of the rectangle image. However, the implementation of Laplace transforms was written with the intent to successfully process any input image. Such intent poses a challenge because each image is different. To overcome this challenge σ , ω , were scaled very closely to $+1.0$ ($0.99\dots$) by means of the sigmoidal function $1.0/(1.0 + e^{-10 \cdot x})$ with $x = \sigma$, $x = \omega$.

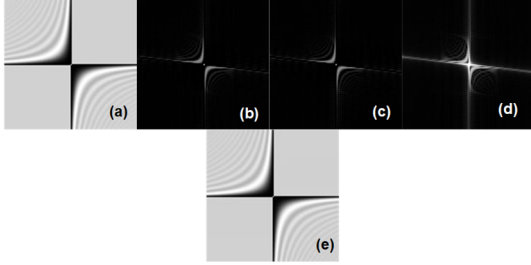


Fig. 2. Sinc signal image (a), the real component of the L-space (b), the imaginary component of the L-space (c), the L-space magnitude (d).

Reconstructed image by inverse Laplace Transformation (e). The warp was obtained scaling the complex variable $s = \sigma + i\omega$ with sigmoidal function. The effect of the warp is to scale both σ and ω to fine fractions of one (1.0).

2.4 Inverse Laplace transformation

The inverse Laplace transformation [24] is

$$f(z) = \int_{-\infty}^{+\infty} f(s) e^{sz} ds \quad (16)$$

Now let us consider that the notation of the j -th component of the L-space is $[f_j(s), R_e] + i[f_j(s), I_m]$ with $j = 1, 2 \dots (N_x \cdot N_y)$.

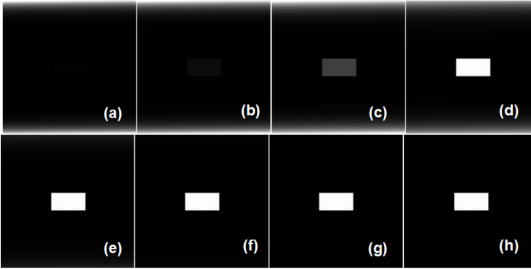


Fig. 3. Rectangle image reconstructed with inverse Laplace transform.

The complex variable $s = \sigma + i\omega$ changes for the value of σ but not the value of ω , which was kept equal to one. (a), (b), (c), (d), (e) show the result of image reconstruction using values of σ for which the inverse Laplace transform fails to reconstruct the rectangle image. The values are: -9.0 (a), -8.0 (b), -7.0 (c), -6.0 (d), -5.0 (e). (f), (g), (h) show successful reconstruction obtained using the values: -4.0 (f), -3.0 (g), -2.0 (h). It was confirmed empirically that with the increase of σ from -0.1 to 1.0 the image is reconstructed. This result means that for the inverse Laplace transform to be successful in reconstructing the rectangle image σ must be greater than -5.0 (see (f)). Now recall that

$$f(z) = \int_{-\infty}^{+\infty} f(s) e^{(\sigma + i\omega)z} ds = \int_{-\infty}^{+\infty} [f(s) e^{\sigma z}] e^{i\omega z} ds \quad (17)$$

Note that $\sigma \cdot z$ can be called ρ , so that we use the position e^ρ . Thus, in view of Eq. (17) we can write

$$(f_p(z), \text{Re}) + i(f_p(z), \text{Im}) \approx \sum_{j=1}^{(N_x \cdot N_y)} [(f_j(s), \text{Re}) \cdot \cos(\text{phase}_{jp}) - (f_j(s), \text{Im}) \cdot \sin(\text{phase}_{jp})] e^\rho + \sum_{j=1}^{(N_x \cdot N_y)} i [(f_j(s), \text{Re}) \cdot \sin(\text{phase}_{jp}) + (f_j(s), \text{Im}) \cdot \cos(\text{phase}_{jp})] e^\rho \quad (18)$$

with $p = 1, 2 \dots (N_x \cdot N_y)$

The value of the phase changes for each (j, p) couple determined by $j = 1, 2 \dots (N_x \cdot N_y)$, and $p = 1, 2 \dots (N_x \cdot N_y)$. The $\approx$ sign applies to the approximation we make when we work with discrete finite signals. Finally, the p -th pixel of the real part of the inverse Laplace reconstructed image is $(f_p(z), \text{Re}) =$

$$\sum_{j=1}^{(N_x \cdot N_y)} [(f_j(s), \text{Re}) \cdot \cos(\text{phase}_{jp}) - (f_j(s), \text{Im}) \cdot \sin(\text{phase}_{jp})] e^\rho \quad (19)$$

with $p = 1, 2 \dots (N_x \cdot N_y)$

And, the p -th pixel of the imaginary part of the inverse Laplace reconstructed image is $(f_p(z), \text{Im}) =$

$$\sum_{j=1}^{(N_x \cdot N_y)} [(f_j(s), \text{Re}) \cdot \sin(\text{phase}_{jp}) + (f_j(s), \text{Im}) \cdot \cos(\text{phase}_{jp})] e^\rho \quad (20)$$

with $p = 1, 2 \dots (N_x \cdot N_y)$

And the magnitude of the reconstructed signal is

$$f_p(z) = \sqrt{(f_p(z), \text{Re})^2 + (f_p(z), \text{Im})^2} \quad (21)$$

with $p = 1, 2 \dots (N_x \cdot N_y)$

2.5 Direct Z transformation

The z-transform of a general sequence $x[n]$ [24] is defined as

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} \quad (22)$$

We express the complex variable z in polar form as Eq. (23), with r as the magnitude of z , and ω as the angle of z . $z = r e^{i\omega}$ (23)

$$X(r e^{i\omega}) = \sum_{n=-\infty}^{+\infty} x[n] (r e^{i\omega})^{-n} = \sum_{n=-\infty}^{+\infty} \{x[n] r^{-n}\} e^{-i\omega n} \quad (24)$$

We see that $X(r e^{i\omega})$ is the Fourier transform of the sequence $x[n]$ multiplied by a real exponential r^{-n} that is

$$X(r e^{i\omega}) = F\{x[n] r^{-n}\} \quad (25)$$

Now let us write Eq. (24) as

$$X(r e^{i\omega}) = \sum_{n=-\infty}^{+\infty} \{x[n] r^{-n}\} e^{-i\omega n} \approx \sum_{n=1}^{(N_x \cdot N_y)} [\text{Re}(n) \cdot \cos(\omega n) + \text{Im}(n) \cdot \sin(\omega n)] r^{-n} + \sum_{n=1}^{(N_x \cdot N_y)} i [-\text{Re}(n) \cdot \sin(\omega n) + \text{Im}(n) \cdot \cos(\omega n)] r^{-n} \quad (26)$$

Moreover, when $\omega n = \text{phase} = 2.0 \pi x y / (N_x \cdot N_y)$; and when the signal is a real number, which means that $\text{Re}(n) = \text{Im}(n) = x[n] = f(n)$; we can write Eq. (26) as $(F_p, \text{Re}) + i (F_p, \text{Im}) \approx$

$$\sum_{n=1}^{(N_x \cdot N_y)} f(n) \cdot [\cos(\text{phase}_{np}) + \sin(\text{phase}_{np})] r^{-n} + \sum_{n=1}^{(N_x \cdot N_y)} i f(n) \cdot [-\sin(\text{phase}_{np}) + \cos(\text{phase}_{np})] r^{-n} \quad (27)$$

with $p = 1, 2 \dots (N_x \cdot N_y)$

The value of the phase changes for each (n, p) couple determined by $n = 1, 2 \dots (N_x \cdot N_y)$, and $p = 1, 2 \dots (N_x \cdot N_y)$. The equation of the p -th pixel of the real component of the z-space is $(F_p, \text{Re}) =$

$$\sum_{n=1}^{(N_x \cdot N_y)} f(n) \cdot [\cos(\text{phase}_{np}) + \sin(\text{phase}_{np})] \cdot r^{-n} \quad (28)$$

with $p = 1, 2 \dots (N_x \cdot N_y)$

The equation of the p -th pixel of the imaginary component of the z-space is $(F_p, \text{Im}) =$

$$\sum_{n=1}^{(N_x \cdot N_y)} f(n) \cdot [-\sin(\text{phase}_{np}) + \cos(\text{phase}_{np})] \cdot r^{-n} \quad (29)$$

with $p = 1, 2 \dots (N_x \cdot N_y)$

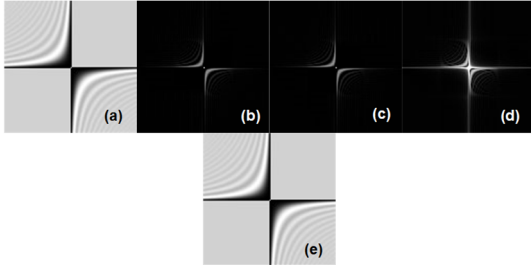


Fig. 4. Sinc signal image (a), the real component of the z-space (b), the imaginary component of the z-space (c), the z-space magnitude (d). Reconstructed image by inverse Z Transformation (e).

(F_p, R_e) is the p -th pixel of the real component image of the z-space (see Figure 4b), (F_p, I_m) is the p -th pixel of the imaginary component image of the z-space (see Figure 4c). Each pixel (F_p, R_e) of the real component image of the z-space is calculated with Eq. (28) with $p = 1, 2 \dots (N_x \cdot N_y)$. Each pixel (F_p, I_m) of the imaginary component image of the z-space is calculated with Eq. (29) with $p = 1, 2 \dots (N_x \cdot N_y)$. The magnitude of the z-space is: $M_p = \sqrt{(F_p, \text{Re})^2 + (F_p, \text{Im})^2}$ with $p = 1, 2 \dots (N_x \cdot N_y)$; (see Figure 4d).

2.6 Inverse Z transformation

The z-space of the signal $x[n]$ is made of a real (F_n, R_e) and an imaginary (F_n, I_m) component, with $n = 1, 2 \dots (N_x \cdot N_y)$.

$N_y)$. Moreover, when $\omega n = \text{phase} = 2.0 \pi x y / (N_x \cdot N_y)$, it follows that

$$(f[s], \text{Re}) + i (f[s], \text{Im}) \approx$$

$$\sum_{n=1}^{(N_x \cdot N_y)} [(F_n, \text{Re}) \cdot \cos(\text{phase}_{ns}) - (F_n, \text{Im}) \cdot \sin(\text{phase}_{ns})] \cdot r^n + \sum_{n=1}^{(N_x \cdot N_y)} i [(F_n, \text{Re}) \cdot \sin(\text{phase}_{ns}) + (F_n, \text{Im}) \cdot \cos(\text{phase}_{ns})] \cdot r^n \quad (30)$$

with $s = 1, 2 \dots (N_x \cdot N_y)$

The value of the phase changes for each (n, s) couple determined by $n = 1, 2 \dots (N_x \cdot N_y)$, and $s = 1, 2 \dots (N_x \cdot N_y)$. Where the s -th pixel of the real component of the reconstructed signal is $(f[s], \text{Re}) \approx$

$$\sum_{n=1}^{(N_x \cdot N_y)} [(F_n, \text{Re}) \cdot \cos(\text{phase}_{ns}) - (F_n, \text{Im}) \cdot \sin(\text{phase}_{ns})] \cdot r^n \quad (31)$$

with $s = 1, 2 \dots (N_x \cdot N_y)$

And the s -th pixel of the imaginary component of the reconstructed signal is $(f[s], \text{Im}) \approx$

$$\sum_{n=1}^{(N_x \cdot N_y)} [(F_n, \text{Re}) \cdot \sin(\text{phase}_{ns}) + (F_n, \text{Im}) \cdot \cos(\text{phase}_{ns})] \cdot r^n \quad (32)$$

with $s = 1, 2 \dots (N_x \cdot N_y)$

And the magnitude of the reconstructed signal is

$$f(s) = \sqrt{(f[s], \text{Re})^2 + (f[s], \text{Im})^2} \quad (33)$$

with $s = 1, 2 \dots (N_x \cdot N_y)$

2.7 Fourier-shift and Z-shift

The Fourier shift theorem states that a shift in phase corresponds to a shift in image space. The shift in phase is a shift in the frequency domain. There are two tasks to perform to implement the phase shift. One is to transform the misplacement in image space into a phase shift in the frequency domain and the other one is to encode the shift in the Fourier formulae. The following implementation formulae are used to transform the misplacement in image space into a phase shift in the frequency domain

$$dx = (i - \frac{N_y}{2}) \quad (34)$$

$$dy = (j - \frac{N_x}{2}) \quad (35)$$

$$x = (dy \cdot N_y + dx) \quad (36)$$

$$y = (dx \cdot N_x + dy) \quad (37)$$

$$xshift = 2.0 \pi y \cdot \frac{x Fshift}{N_x} \quad (38)$$

$$yshift = 2.0 \pi x \cdot \frac{y Fshift}{N_y} \quad (39)$$

Where i, j, dx, dy are the pixel identifiers (integer numbers), with $i = 1, 2 \dots N_x$ (number of pixels along the X coordinate) and $j = 1, 2 \dots N_y$ (number of pixels along the Y coordinate). And, x and y identify the pixel's coordinate. $x Fshift$ and $y Fshift$ are values of the image space misplacement, and $xshift$ and $yshift$ are the phase shifts. The phase shifts $xshift$ and $yshift$ embed the image space shifts $x Fshift$ and $y Fshift$. The second challenge is to encode the shift in the phase as follows.

$$phase = \frac{(2.0 \pi \cdot k_2 \cdot k_3)}{(N_x \cdot N_y)} \quad (40)$$

$$phase += xshift + yshift \quad (41)$$

k_2 and k_3 were defined in section 2.1 titled 'Direct Fourier transformation'. Once we sum the phase shifts to the phase, we have encoded the misplacement in Fourier formulae. Similarly, the Z-transform allows the shift in image space to be encoded in the z-space and this happens through the exponent n . The image space shifts $xFshift$ and $yFshift$ are encoded using the following formulae

$$xshiftExponent = 2.0 \pi y \cdot \frac{xFshift}{N_x} \quad (42)$$

$$yshiftExponent = 2.0 \pi x \cdot \frac{yFshift}{N_y} \quad (43)$$

The phase then becomes

$$phase += xshiftExponent + yshiftExponent \quad (44)$$

Figure 5 shows an illustration of the Fourier shift theorem and the effect of phase-shifting with the Z-transform so to obtain image space shift after image reconstruction with inverse Z-transform.

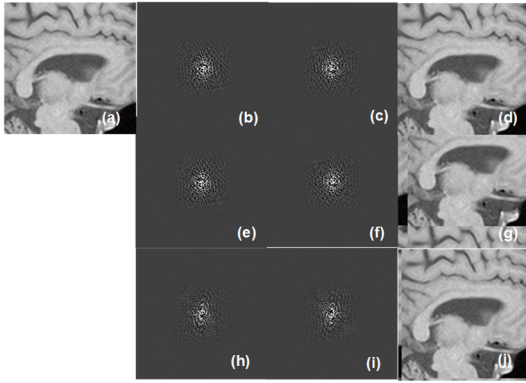


Fig. 5. Magnetic Resonance Image (a). The real component of the k-space (b). The imaginary component of the k-space (c).

The image was reconstructed with inverse Fourier transform (d). The real component (e) and the imaginary component (f) of the k-space encoding an image space shift of $(x, y) = (13, 31)$ pixels. The reconstructed image (g) with inverse Fourier transform using (e) and (f), and which shows the image space shift. The real component (h) and the imaginary component (i) of the z-space encoding an image space shift of $(x, y) = (4, 4)$ pixels. The reconstructed image (j) with inverse Z transform using (h) and (i), and which shows the image space shift.

3. Results

The dataset used to validate software implementation of theoretical concepts comprises of 30 two-dimensional images: 10 theoretical images and 20 Magnetic Resonance (MR) images. The results reported in this section are a sample of the validation performed using the dataset.

3.1 Maximum reconstruction error

The image reconstructed with the inverse Fourier transform is subject to a scaling factor that adjusts the pixel

intensity values to match the values of the departing image. The scaling factor is constant across pixels. After reconstruction, the difference image is calculated by subtracting the reconstructed image from the departing image. The maximum reconstruction error is the maximum pixel intensity absolute value of the difference image. As extension to earlier work [25], a calibration study was developed so to determine the evolution of the maximum reconstruction error (inverse Fourier transform) versus the scale factor (see Figure 6b).

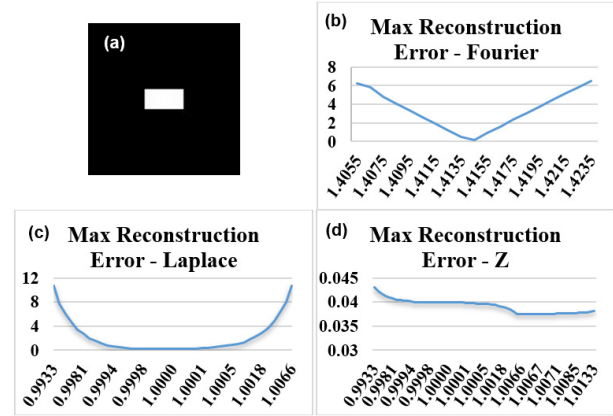


Fig. 6. Rectangle image (a). (b) Evolution of the maximum reconstruction error of the inverse Fourier transform versus the scale factor (the minimum is 0.203 with a scaling factor equal to 1.4145). (c) Evolution of the maximum reconstruction error of the inverse Laplace transform versus the complex variable (the minimum is 0.199 for $s = 0.999864 + i 0.999864$). (d) Evolution of the maximum reconstruction error of the inverse Z transform versus the complex variable (the minimum is 0.0375 for $z = 1.006989 + i 1.006989$).

The rectangle image showed in Figure 6a was used in this study. The plot in Figure 6b reveals that the minimum of the maximum error is 0.203 for the value of the scaling factor equal to 1.4145. This value of the scaling factor was used to determine the evolution of the maximum reconstruction error of Laplace transform (see Figure 6c) and Z transform (see Figure 6d) versus the value of the complex variable. Real and imaginary components of the complex variable had the same numerical value (plotted along the abscissa in (c) and (d)). The minimum of the maximum reconstruction error was: 0.199 for the Laplace transform with $s = 0.999864 + i 0.999864$; and 0.0375 for the Z transform with $z = 1.006989 + i 1.006989$.

3.2 Laplace and Z transforms: coherence to Fourier theory

This section presents results that show that both Laplace and Z transforms are the same as the Fourier transform when the following conditions are met.

1. Laplace transform. The complex variable is $s = \sigma + i \omega$, and so when $s = i \omega$, which means when $\sigma = 0$, the Laplace transform is the Fourier transform. This case is presented in Figure 7.

2. Z transform. The complex variable z in polar form is $z = r e^{j\omega}$, and so when $r = 1$, the Z transform is the Fourier transform. This case is also presented in Figure 7.

Points 1 and 2 are well-known, and fundamental and they offer the opportunity to validate the implementation of direct and inverse Fourier, Laplace, and Z transforms. Figure 7 shows that k-space, L-space, and z-space are identical, and this is evidence that the three direct transforms were designed with the correct theoretical background presented in the theory section and were also correctly implemented into software. Figure 7 also shows that images reconstructed through inverse transformations (d), (g), (j) are the same as (a).

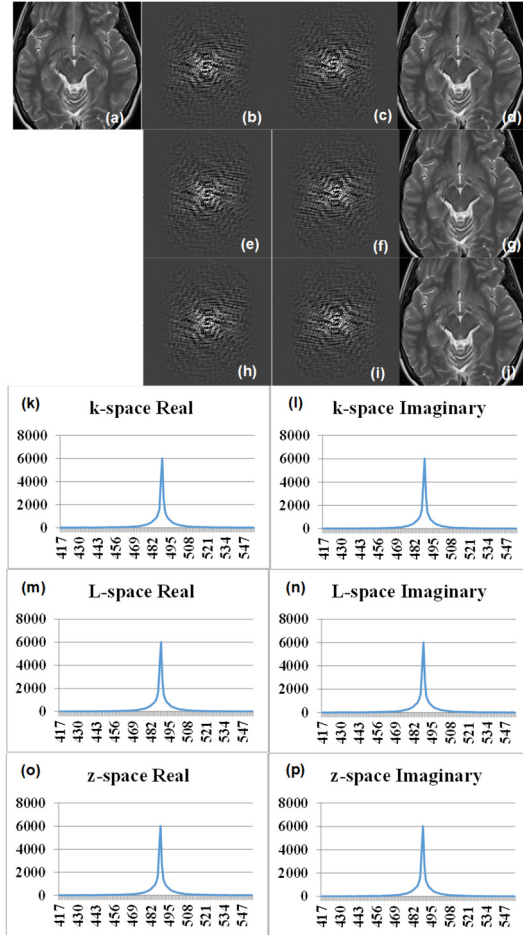


Fig. 7. MR image (a). Fourier k-space real component (b), Fourier k-space imaginary component (c), reconstructed signal through inverse Fourier transformation (d). Laplace L-space real component (e), Laplace L-space imaginary component (f), reconstructed signal through inverse Laplace transformation (g). Z-space real component (h), z-space imaginary component (i), reconstructed signal through inverse Z transformation (j). (k), (l), (m), (n), (o), (p), Histograms of the images in (b), (c), (e), (h), (f), (i), respectively.

3.3 Laplace and Z transforms versus Fourier transform

This section presents results that show the comparison of both Laplace and Z transforms versus the Fourier transform (see Figures 8, 9). The assumption is that if the

inverse transformation yields to the departing image, then the direct transformation is correct, and so is the inverse transformation. These results were obtained using $\sigma \neq 0$ and $r \neq 1$. The histograms in Figures 8, 9 show that k-space and L-space differs, and L-space and z-space also differs, both in their real and imaginary components. In Figure 8, the complex variable of the Laplace transform is $s = \sigma + i\omega = 0.999877 + i 0.999877$; and the complex variable of the Z transform is $z = (R_e + i I_m) = 0.999877 + i 0.999877$. In Figure 9, $s = 0.999665 + i 0.999665$; $z = 0.999665 + i 0.999665$.

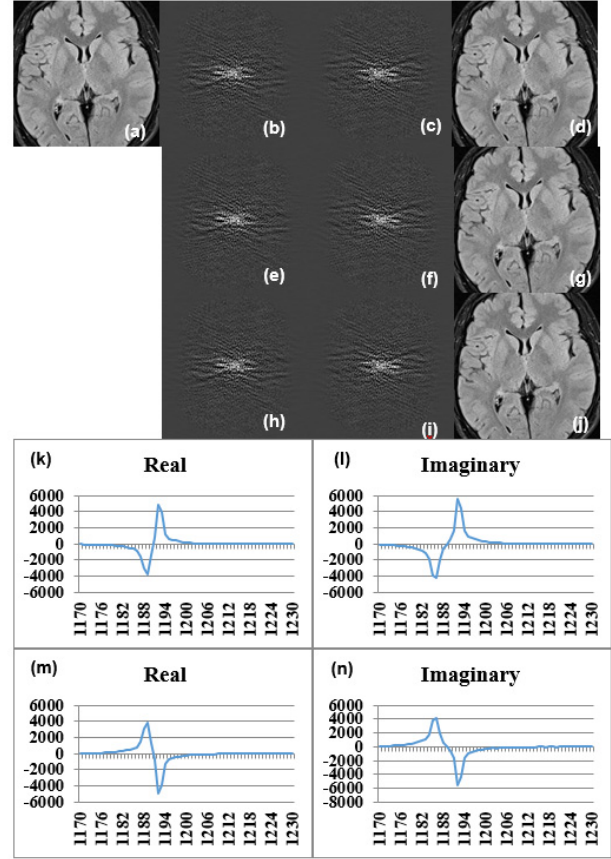


Fig. 8. MR image (a). Fourier k-space real component (b), Fourier k-space imaginary component (c), reconstructed signal through inverse Fourier transformation (d). Laplace L-space real component (e), Laplace L-space imaginary component (f), reconstructed signal through inverse Laplace transformation (g). Z-space real component (h), z-space imaginary component (i), reconstructed signal through inverse Z transformation (j). (k) Difference between the histogram of k-space real (b) and the histogram of L-space real (e). (l) Difference between the histogram of k-space imaginary (c) and the histogram of L-space imaginary (f). (m) Difference between the histogram of L-space real (b) and the histogram of z-space real (h). (n) Difference between the histogram of L-space imaginary (c) and the histogram of z-space imaginary (i).

3.4 Comparison of Fourier-shift and Z-shift transforms

Another well-known and fundamental property of Fourier theory is the effect of a phase shift in the frequency

domain. Such effect is visible as an image space shift after inverse transformation.

The verification of this property through implementation was adopted as test paradigm of the theoretical background presented in section: ‘2.7 Fourier-shift and z-shift’; and it is presented in Figure 10d, which shows the result of inverse Fourier transformation; and in (g), (j), which show the results of inverse Z transformation.

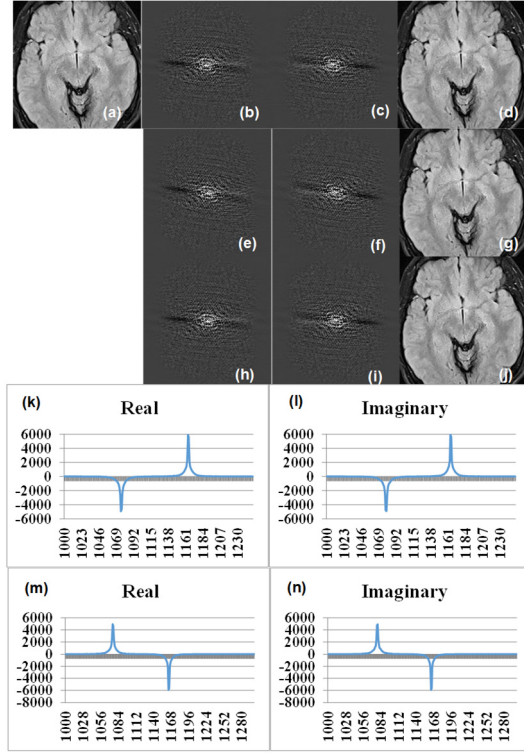


Fig. 9. MR image (a). Fourier k-space real component (b), Fourier k-space imaginary component (c), reconstructed signal through inverse Fourier transformation (d). Laplace L-space real component (e), Laplace L-space imaginary component (f), reconstructed signal through inverse Laplace transformation (g). Z-space real component (h), z-space imaginary component (i), reconstructed signal through inverse Z transformation (j). (k) Difference between the histograms of k-space real (b) and L-space real (e). (l) Difference between the histogram of k-space imaginary (c) and the histogram of L-space imaginary (f). (m) Difference between the histogram of L-space real (b) and the histogram of z-space real (h). (n) Difference between the histogram of L-space imaginary (c) and the histogram of z-space imaginary (i).

Figure 10 also shows that when $r = 1$ for the z transform, Fourier-shift and z-shift yields to same frequency domain (see k-space in Figures 10b, 10c, and z-space in Figures 10e, 10f) and same reconstructed image (see Figures 10d, 10g). Figure 11 extends the analysis of Figure 10, presenting histograms of k-space images and z-space images seen in Figure 10. As visible in Figures 11a, 11b (histograms of k-space) and Figures 11c, 11d (histograms of z-space), the frequency domain is the same. Figures 12, and 13 shows the case of a T2 MRI processed with Fourier-shift and z-shift when $r = 1$. The frequency domain encodes the same image space shift for the two

transforms (various magnitude of the shift as quantifiable in pixels along the two Cartesian coordinates). K-space and z-space were the same (not shown) and so were the reconstructed MRIs (see Figures 12, and 13). As far as regards the comparison between the z-space when $r = 1$ (Figures 10e, 10f) and when $r \neq 1$ (Figures 10h, 10i; $z = (Re + i Im) = 0.998178 + i 0.998178$); the histograms presented in Figures 11e, 11f reveal the difference.

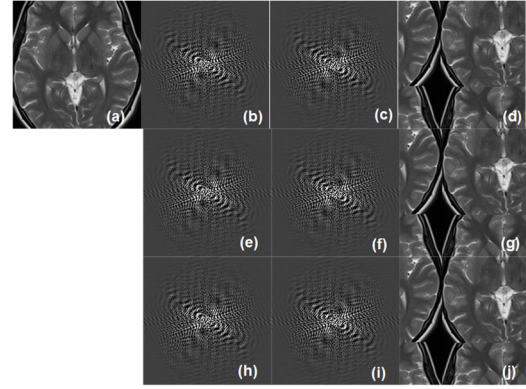


Fig. 10. MR image in (a). Fourier k-space real component (b), Fourier k-space imaginary component (c), reconstructed signal through inverse Fourier transformation (d). In (b), (c) the k-space encodes an image space shift of 60 pixels in the x direction and 60 pixels in the y direction. In (d) the shift is reconstructed in the image space. Z-space real component (e), z-space imaginary component (f), reconstructed signal through inverse Z transformation (g). In (e), (f), (h), (i) the z-space encodes an image space shift of 60 pixels in the x direction and 60 pixels in the y direction. In (g), (j) the shift is reconstructed in the image space. In (e), (f) the complex variable is $z = r e^{j\omega}$ (with $r = 1$). In (h), (i) the complex variable is $z = (Re + i Im) = 0.5 + i 0.5$.

In summary, Figures 10, 11, 12, 13, present three main results: 1. The effect of a phase shift in the frequency domain is an image space shift visible through Fourier and Z inverse transformations (Figures 10d, 10g, 10j, 12, and 13). 2. When $r = 1$, and when the frequency domain encodes the same image space shift, z-space and k-space are the same (see Figures 11a, 11b, 11c, 11d). 3. The evidence that when $r = 1$ and $r \neq 1$, the z-space encoding the same image space shift (60 pixels along x and y directions) is not the same (see histograms in Figures 11e, 11f).

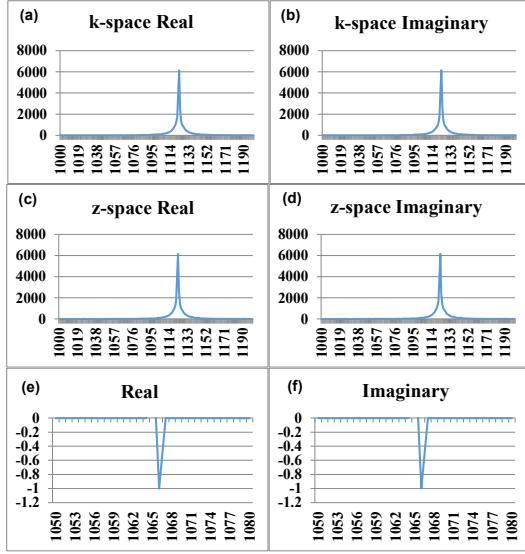


Fig. 11. (a), (b) Histograms of the real and imaginary components of k-space presented in Figures 10b, 10c, respectively. (c), (d) Histograms of the real and imaginary components of z-space presented in Figures 10e, 10f, respectively. The complex variable is $z = r e^{j\omega}$ (with $r = 1$). Hence, k-space and z-space are the same as visible in Figures 10b, 10c, 10e, 10f, and in the histograms. (e) Difference between the histogram of z-space real when $r = 1$ (see Figure 10e) and the histogram of z-space real when $z = (Re + i Im) = 0.5 + i 0.5$ (see Figure 10h). (f) Difference between the histogram of z-space imaginary when $r = 1$ (see Figure 10f) and the histogram of z-space imaginary when $z = (Re + i Im) = 0.5 + i 0.5$ (see Figure 10i).

Figures 12 and 13 show the effect of the Fourier and Z shifts (with $r = 1$) on a T2 MRI. The magnitude of the shift (measured in pixels) changes at each experimental session and takes the following values: $(-33, -33)$, $(-30, -30)$, $(-27, -27)$, $(-24, -24)$, $(-21, -21)$, $(-18, -18)$, $(-15, -15)$, $(-12, -12)$, $(-9, -9)$, $(-6, -6)$, $(5, 5)$, $(8, 8)$, $(11, 11)$, $(13, 13)$, $(16, 16)$, $(20, 20)$, $(23, 23)$, $(26, 26)$, $(29, 29)$, $(32, 32)$. Through exact inverse image reconstruction, Figures 12, and 13, reinforce on congruency of transforms, and coherence between transforms.

4. Discussion

4.1 Recent trends in Fourier theory & validation procedure

In recent years computation of fractional Fourier transform (FRFT) has been under the attention of the scientific community and FRFT was demonstrated to be an extension of the classic Fourier transform [26] through a parametrization that makes the FRFT a framework encompassing FT and inverse FT too. A review of existing methods lists a survey of FRFT applications and provides a listing of existing software for FRFT calculation [27]. Cornelius Lanczos did pioneer work on the Fast Fourier transform (FFT), while the pioneer paper was published by Cooley & Tukey in 1965 [1], and the most comprehensive FFT implementation is currently recognized to be the one reported by Frigo & Johnson [28].

However, with the advent of big data disciplines, sparse Fourier transform (SFT) has become a recent trend. SFT allows the calculation of the Fourier transform in a short time using a subset of data, hence focusing on the non-zero frequencies [29]. Recent applications of Fourier theory include: 1. Approximation of Gaussian filters through the Fourier transform [30]; 2. Fourier burst accumulation, which removes image blurring by weighted average in the Fourier domain when the blur is time-invariant [31, 32]. 3. Non-linear Fourier transform which decomposes a periodic signal into nonlinearly interacting waves [33] and finds applications in fiber optics processing and transmission [34]. 4. Non-uniform sampling methods in Nuclear Magnetic Resonance imaging [35], or fast Fourier transform (FFT) methods based on non-equally spaced nodes located on polar and pseudo-polar grids [36]. 5. Calculation of probability cumulative distribution functions through inverse Laplace transforms [37-39]. 6. Wiener-Hopf factorization of a complex function using Hilbert and Z-transforms [40-42]. 7. Improved convergence of Fourier, Hilbert, and Z-transforms [41]. 8. Inverse Fourier transform-based MRI k-space filtering techniques using the intensity-curvature functional [43].

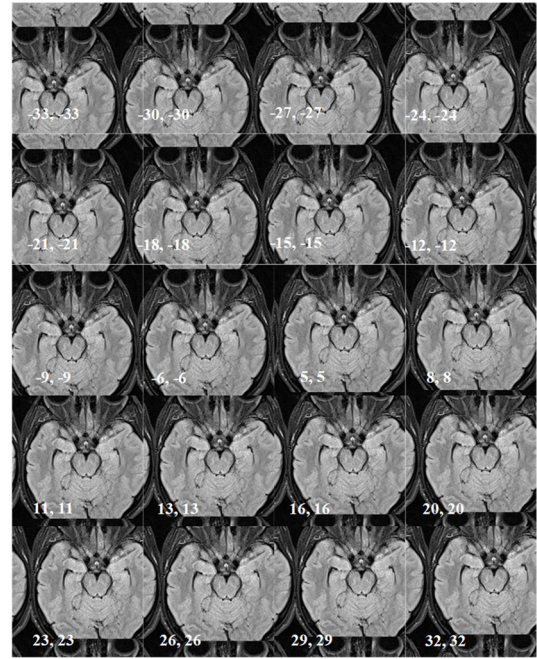


Fig. 12. Reconstructed MRIs with inverse Fourier transform. For each of the cases presented in this figure, the frequency domain encodes a phase shift, which is reconstructed as image space shift. The shifts are measured in pixels (see white labels) along x and y directions.

The present research provides both theoretical and applied aspects along with the most significant effort which is the validation procedure. Validation of the code was performed using two approaches. The first approach looks at the intersections between: 1. Fourier and Laplace transforms; 2. Fourier and Z transforms. When the value of the complex variable of Laplace transform is $s = \sigma + i\omega$ with $s = i\omega$ and $\sigma = 0$; the Laplace transform is the

Fourier transform. When the value of the complex variable of Z transform is $z = r e^{j\omega}$ with $r = 1$; the Z transform is the Fourier transform too. It follows that for $\sigma = 0$ and $r = 1$ k-space, L-space and z-space are the same. This was documented in Figure 7 and extensively observed across the whole dataset of 30 images. The second approach uses the inverse transform to validate the direct transform. The latter was used to validate separately Fourier, Laplace, and Z transforms. Implicitly, this approach yields: 1. to confirm that direct and inverse transforms are correct; and 2. to investigate the difference between k-space, L-space, and z-space when $\sigma \neq 0$ and $r \neq 1$. This second result is documented in Figures 8, 9 and it was confirmed across the test set of 30 images.



Fig. 13. Reconstructed MRIs with inverse Z transform. The frequency domain encodes the same phase shifts as per Figure 12. The phase shifts are reconstructed as image space shifts exactly as per Fourier shifts because the magnitude of the complex number in the Z-transform is $r = 1$.

5. Conclusion

This research reports three main results obtained studying direct and inverse Fourier, Laplace, and Z transformations: 1. theory, 2. implementation, and 3. testing. The theoretical background is reported as a study guide. Software implementation was accomplished in C++. Testing was performed with two-dimensional images by seeking for coherence and congruency among the three transforms. Coherence between the software implementation of the three transformations is investigated when: 1. the real component σ of the complex variable ($s = \sigma + i\omega$) is equal to zero, which is the case when Fourier and Laplace transforms are the same; and 2. When the magnitude r of the complex variable ($z = r e^{j\omega}$) is equal to one, which is the case when Fourier and Z transforms are the same. Congruency between software implementation of transformations is assessed comparing departing image and inverse reconstructed image.

Hence, the contribution of this research is within the following three areas.

1. The theoretical background presentation style. Presentation of the concepts is given to simplify the notation, include details, and make possible software implementation of Fourier, Laplace, and Z direct and inverse transforms.

2. Validation paradigm used: (a) to test the coherence of Fourier transforms with Laplace and Z transforms; (b) to compare the frequency domain of Fourier, Laplace, and Z transforms; (c) to compare frequency shifts (which effect is to create image space shifts) between Fourier and Z transforms.

The novelty of this paper consists of details of theory that are not discernible reading the literature. Hence, the focus of the paper is to provide well-tested mathematical guidelines for the implementation of direct and inverse Fourier, Laplace, and Z transformations. Literature on these three transforms is huge, nonetheless, mathematical details are not clearly stated, because of the presentation of advanced concepts, which take basic concepts for granted.

Acknowledgements

The author expresses sincere gratitude to Dr. Dimitar Veljanovski and Dr. Filip A. Risteski because of the availability of the MR images and to Professor Ustijana Rechkoska Shikoska for the help provided to coordinate the human resources. Dr. Dimitar Veljanovski and Dr. Filip A. Risteski are affiliated with General Hospital 8-mi Septemvri, Skopje, R. N. Macedonia. Professor Ustijana Rechkoska Shikoska is the Vice Rector of the University of Information Science and Technology, Ohrid, R. N. Macedonia.

References

- [1] Cooley, J. W. and Tukey, J. W. (1965). An algorithm for the machine calculation of complex Fourier series, *Mathematics of Computation*, vol. 19, no. 90, 297-301.
- [2] Cooley, J. W., Lewis, P. A. and Welch, P. D. (1967). Historical notes on the fast Fourier transform, *Proceedings of the IEEE*, vol. 55, no. 10, pp. 1675-1677.
- [3] Cooley, J. W., Lewis, P. A. and Welch, P. D. (1969). The fast Fourier transform and its applications, *IEEE Transactions on Education*, vol. 12, no. 1, pp.27-34.
- [4] Robinson, E. A. (1982). A historical perspective of spectrum estimation, *Proceedings of the IEEE*, vol. 70, no. 9, pp.885-907.
- [5] Dubner, H. and Abate, J. (1968). Numerical inversion of Laplace transforms by relating them to the finite Fourier cosine transform, *Journal of the ACM*, vol. 15, no. 1, pp.115-123.
- [6] Stehfest, H. (1970). Algorithm 368: Numerical inversion of Laplace transforms [D5], *Communications of the ACM*, vol. 13, no. 1, pp.47-49.
- [7] Mansfield, P. (1962). Proton magnetic resonance relaxation in solids by transient methods, *Doctoral dissertation*, Queen Mary, University of London. London, U.K.
- [8] Lauterbur, P. C. (1973). Image formation by induced local interactions: examples employing nuclear magnetic resonance, *Nature*, vol. 242, no. 5394, pp.190-191.

- [9] Weeks, W. T. (1966). Numerical inversion of Laplace transforms using Laguerre functions, *Journal of the ACM*, vol. 13, no. 3, pp.419-429.
- [10] Talbot, A. (1979). The accurate numerical inversion of Laplace transforms, *IMA Journal of Applied Mathematics*, vol. 23, no. 1, pp.97-120.
- [11] Garbow, B. S., Giunta, G., Lyness, J. N. and Murli, A. (1988). Software for an implementation of Weeks' method for the inverse Laplace transform, *ACM Transactions on Mathematical Software (TOMS)*, vol. 14, no. 2, pp.163-170.
- [12] Murli, A. and Rizzardi, M. (1990). Algorithm 682: Talbot's method of the Laplace inversion problems, *ACM Transactions on Mathematical Software (TOMS)*, vol. 16, no. 2, pp.158-168.
- [13] D'amore, L., Laccetti, G. and Murli, A. (1999). An implementation of a Fourier series method for the numerical inversion of the Laplace transform, *ACM Transactions on Mathematical Software (TOMS)*, vol. 25, no. 3, pp.279-305.
- [14] Kuhlman, K. L. (2013). Review of inverse Laplace transform algorithms for Laplace-space numerical approaches, *Numerical Algorithms*, vol. 63, no. 2, pp.339-355.
- [15] Khan, M. and Hussain, M. (2011). Application of Laplace decomposition method on semi-infinite domain, *Numerical Algorithms*, vol. 56, no. 2, pp.211-218.
- [16] Kexue, L. and Jigen, P. (2011). Laplace transform and fractional differential equations, *Applied Mathematics Letters*, vol. 24, no. 12, pp.2019-2023.
- [17] Ragazzini, J. R. and Zadeh, L. A. (1952). The analysis of sampled-data systems, *Transactions of the American Institute of Electrical Engineers, Part II: Applications and Industry*, vol. 71, no. 5, pp.225-234.
- [18] Jury, E.I. (1973). Theory application of the z-transform method, R.E. Krieger Pub. Co., Huntington, N.Y., U.S.A.
- [19] Fazarinc, Z. (2013). Z-transform and its application to development of scientific simulation algorithms, *Computer Applications in Engineering Education*, vol. 21, no. 1, pp.75-88.
- [20] Caldarola, F., Maiolo, M. and Solferino, V. (2020). A new approach to the Z-transform through infinite computation, *Communications in Nonlinear Science and Numerical Simulation*, vol. 82, 105019.
- [21] Ozaktas, H. M. and Kutay, M. A. (2001). The fractional Fourier transform, In: *2001 IEEE European Control Conference (ECC)*, pp.1477-1483.
- [22] Hassanzadeh, H. and Pooladi-Darvish, M. (2007). Comparison of different numerical Laplace inversion methods for engineering applications, *Applied Mathematics and Computation*, vol. 189, 2, 1966-1981.
- [23] Rabiner, L., Schafer, R. W. and Rader, C. (1969). The chirp z-transform algorithm, *IEEE Transactions on Audio and Electroacoustics*, vol. 17, no. 2, pp.86-92.
- [24] Oppenheim, A.V., Wilsky, A.S. and Nawab, S.H. (1997) *Signals & Systems*, 2nd Edition, Prentice Hall, Upper Saddle River, N.J., U.S.A.
- [25] Kumi, K., Berisha, R., Biçaku, X., Uruçi, R. and Ciulla, C. (2022). Verification of congruency and coherence of Fourier, Laplace, and Z-transformations, *Int. J. Student Project Reporting* – forthcoming.
- [26] Zayed, A. I. (1996). On the relationship between the Fourier and fractional Fourier transforms, *IEEE Signal Processing Letters*, vol. 3 no. 12, pp.310-311.
- [27] Zhang, Y., Wang, S., Yang, J. F., Zhang, Z., Phillips, P., Sun, P. and Yan, J. (2017). A comprehensive survey on fractional Fourier transform, *Fundamenta Informaticae*, vol. 151 no. (1-4), pp.1-48.
- [28] Frigo, M. and Johnson, S. G. (1998). FFTW: An adaptive software architecture for the FFT, In: *Proceedings of the 1998 IEEE International Conference on Acoustics, Speech and Signal Processing*, ICASSP'98, vol. 3, pp.1381-1384.
- [29] Gilbert, A. C., Indyk, P., Iwen, M. and Schmidt, L. (2014). Recent developments in the sparse Fourier transform: A compressed Fourier transform for big data, *IEEE Signal Processing Magazine*, vol. 31, no. 5, pp.91-100.
- [30] Nair, P., Popli, A. and Chaudhury, K. N. (2017). A fast approximation of the bilateral filter using the discrete Fourier transform, *Image Processing On Line*, vol. 7, pp.115-130.
- [31] Anger, J. and Meinhardt-Llopis, E. (2017). Implementation of local Fourier burst accumulation for video deblurring, *Image Processing On Line*, vol 7, pp.56-64.
- [32] Delbracio, M. and Sapiro, G. (2015). Removing camera shake via weighted Fourier burst accumulation, *IEEE Transactions on Image Processing*, vol. 24, no. 11, pp.3293-3307.
- [33] Wahls, S. and Poor, H. V. (2015). Fast numerical nonlinear Fourier transforms, *IEEE Transactions on Information Theory*, vol. 61, no. 12, pp.6957-6974.
- [34] Turitsyn, S. K., Prilepsky, J. E., Le, S. T., Wahls, S., Frumin, L. L., Kamalian, M. and Derevyanko, S. A. (2017). Nonlinear Fourier transform for optical data processing and transmission: advances and perspectives, *Optica*, vol. 4, no. 3, pp.307-322.
- [35] Kazimierczuk, K. and Orekhov, V. (2015). Non uniform sampling: post Fourier era of NMR data collection and processing, *Magnetic Resonance in Chemistry*, vol. 53 no. 11, pp.921-926.
- [36] Fenn, M., Kunis, S. and Potts, D. (2007). On the computation of the polar FFT, *Applied and Computational Harmonic Analysis*, vol. 22, no. 2, pp.257-263.
- [37] Abate, J. and Whitt, W. (1992). The Fourier-series method for inverting transforms of probability distributions, *Queueing Systems*, vol. 10, no. 1, pp.5-87.
- [38] Abate, J. and Whitt, W. (1992). Numerical inversion of probability generating functions, *Operations Research Letters*, vol. 12, no. 4, pp.245-251.
- [39] Abate, J. and Whitt, W. (1995). Numerical inversion of Laplace transforms of probability distributions, *ORSA Journal on Computing*, vol. 7, no. 1, pp.36-43.
- [40] Fusai, G., Germano, G. and Marazzina, D. (2016). Spitzer identity, Wiener-Hopf factorization and pricing of discretely monitored exotic options, *European Journal of Operational Research*, vol. 251, no. 1, pp.124-134.
- [41] Phelan, C. E., Marazzina, D., Fusai, G. and Germano, G. (2018). Fluctuation identities with continuous monitoring and their application to the pricing of barrier options, *European Journal of Operational Research*, vol. 271, no. 1, pp.210-223.

[42] Phelan, C. E., Marazzina, D., Fusai, G. and Germano, G. (2019). Hilbert transform, spectral filters and option pricing, *Annals of Operations Research*, vol. 282, no. 1, pp.273-298.

[43] Ciulla, C. (2021). Inverse Fourier transformation of combined first order derivative and intensity-curvature functional of magnetic resonance angiography of the human brain, *Computer Methods and Programs in Biomedicine*, vol. 211, 106384.

Appendix

1. Direct process

$$e^{-2\pi i \varphi z} = \cos(-2\pi \varphi z) + i \sin(-2\pi \varphi z) =$$

$$\cos(2\pi \varphi z) - i \sin(2\pi \varphi z) \quad (1)$$

$$f(z) = [\operatorname{Re}(z) + i \operatorname{Im}(z)] \quad (2)$$

$$[\operatorname{Re}(z) + i \operatorname{Im}(z)] \cdot [\cos(2\pi \varphi z) - i \sin(2\pi \varphi z)] =$$

$$\operatorname{Re}(z) \cdot \cos(2\pi \varphi z) - i (\operatorname{Re}(z) \cdot \sin(2\pi \varphi z)) +$$

$$i (\operatorname{Im}(z) \cdot \cos(2\pi \varphi z)) + (\operatorname{Im}(z) \cdot \sin(2\pi \varphi z)) =$$

$$(\operatorname{Re}(z) \cdot \cos(2\pi \varphi z)) + (\operatorname{Im}(z) \cdot \sin(2\pi \varphi z)) +$$

$$i [(-\operatorname{Re}(z) \cdot \sin(2\pi \varphi z)) + (\operatorname{Im}(z) \cdot \cos(2\pi \varphi z))] =$$

$$f(z) \cdot e^{-2\pi i \varphi z} \quad (3)$$

2. Inverse process

$$e^{2\pi i \varphi z} = \cos(2\pi \varphi z) + i \sin(2\pi \varphi z) =$$

$$\cos(2\pi \varphi z) + i \sin(2\pi \varphi z) \quad (4)$$

$$f(\varphi) = [(F_k, \operatorname{Re}) + i (F_k, \operatorname{Im})]; k = 1, 2 \dots (N_x \cdot N_y) \quad (5)$$

$$f(\varphi) \cdot e^{2\pi i \varphi z} =$$

$$[(F_k, \operatorname{Re}) + i (F_k, \operatorname{Im})] \cdot [\cos(2\pi \varphi z) + i \sin(2\pi \varphi z)] =$$

$$(F_k, \operatorname{Re}) \cdot \cos(2\pi \varphi z) + i (F_k, \operatorname{Re}) \cdot \sin(2\pi \varphi z) +$$

$$i (F_k, \operatorname{Im}) \cdot \cos(2\pi \varphi z) - (F_k, \operatorname{Im}) \cdot \sin(2\pi \varphi z) =$$

$$(F_k, \operatorname{Re}) \cdot \cos(2\pi \varphi z) - (F_k, \operatorname{Im}) \cdot \sin(2\pi \varphi z) +$$

$$i [(F_k, \operatorname{Re}) \cdot \sin(2\pi \varphi z) + (F_k, \operatorname{Im}) \cdot \cos(2\pi \varphi z)] \quad (6)$$